Problem 1. Find all prime numbers a, b, c and positive integers k which satisfy the equation

$$a^2+b^2+16\cdot c^2=9\cdot k^2+1.$$

Solution:

The relation $9\cdot k^2+1\equiv 1 \pmod{3}$ implies

$$a^2+b^2+16\cdot c^2\equiv 1 \pmod 3 \iff a^2+b^2+c^2\equiv 1 \pmod 3.$$

Since $a^2\equiv 0,1 \pmod 3$, $b^2\equiv 0,\ 1 \pmod 3$, $c^2\equiv 0,\ 1 \pmod 3$, we have:

a^2	0	0	0	0	1	1	1	1
b^2	0	0	1	1	0	0	1	1
c^2	0	1	0	1	0	1	0	1
$a^2+b^2+c^2$	0	1	1	2	1	2	2	0

From the previous table it follows that two of three prime numbers $a,\ b,\ c$ are equal to 3.

Case 1. $a=b=3$

We have

$$a^2+b^2+16\cdot c^2=9\cdot k^2+1 \iff 9\cdot k^2-16\cdot c^2=17 \iff (3k-4c)\cdot(3k+4c)=17,$$

$$\iff \begin{cases}3k-4c=1,\\ 3k+4c=17,\end{cases} \iff \begin{cases}c=2,\\ k=3,\end{cases} \text{ and } (a,b,c,k)=(3,3,2,3).$$

Case 2. $c=3$

If $(3,b_0,c,k)$ is a solution of the given equation, then $(b_0,3,c,k)$ is a solution too.

Let $a=3$. We have

$$a^2+b^2+16\cdot c^2=9\cdot k^2+1 \iff 9\cdot k^2-b^2=152 \iff (3k-b)\cdot(3k+b)=152.$$

Both factors shall have the same parity and we obtain only 2 cases:

- $\begin{cases}3k-b=2,\\ 3k+b=76,\end{cases} \iff \begin{cases}b=37,\\ k=13,\end{cases}$ and $(a,b,c,k)=(3,37,3,13)$;
- $\begin{cases}3k-b=4,\\ 3k+b=38,\end{cases} \iff \begin{cases}b=17,\\ k=7,\end{cases}$ and $(a,b,c,k)=(3,17,3,7)$.

So, the given equation has 5 solutions:

$$\{(37,3,3,13),\ (17,3,3,7),\ (3,37,3,13),\ (3,17,3,7),\ (3,3,2,3)\}.$$

Problem 2. Let a,b,c be positive real numbers such that $a+b+c=3$. Find the minimum value of the expression

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}.$$

Solution:
We can rewrite A as follows:

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-a^2-b^2-c^2=$$

$$2\left(\frac{ab+bc+ca}{abc}\right)-(a^2+b^2+c^2)=2\left(\frac{ab+bc+ca}{abc}\right)-((a+b+c)^2-2(ab+bc+ca))=$$

$$2\left(\frac{ab+bc+ca}{abc}\right)-(9-2(ab+bc+ca))=2\left(\frac{ab+bc+ca}{abc}\right)+2(ab+bc+ca)-9=$$

$$2(ab+bc+ca)\left(\frac{1}{abc}+1\right)-9.$$

Recall now the well-known inequality $(x+y+z)^2\geq 3(xy+yz+zx)$ and set $x=ab, y=bc, z=ca$, to obtain $(ab+bc+ca)^2\geq 3abc(a+b+c)=9abc$, where we have used $a+b+c=3$. By taking the square roots on both sides of the last one we obtain:

$$ab+bc+ca\geq 3\sqrt{abc}\,. \qquad (1)$$

Also by using AM-GM inequality we get that

$$\frac{1}{abc}+1\geq 2\sqrt{\frac{1}{abc}}\,. \qquad (2)$$

Multiplication of (1) and (2) gives:

$$(ab+bc+ca)\left(\frac{1}{abc}+1\right)\geq 3\sqrt{abc}\cdot 2\sqrt{\frac{1}{abc}}=6\,.$$

So $A\geq 2\cdot 6-9=3$ and the equality holds if and only if $a=b=c=1$, so the minimum value is 3.

Problem 3. Let ΔABC be an acute triangle. The lines l_1, l_2 are perpendicular to AB at the points A, B respectively. The perpendicular lines from the midpoint M of AB to the sides of the triangle AC, BC intersect the lines l_1, l_2 at the points E, F, respectively. If D is the intersection point of EF and MC, prove that

$$\angle ADB=\angle EMF.$$

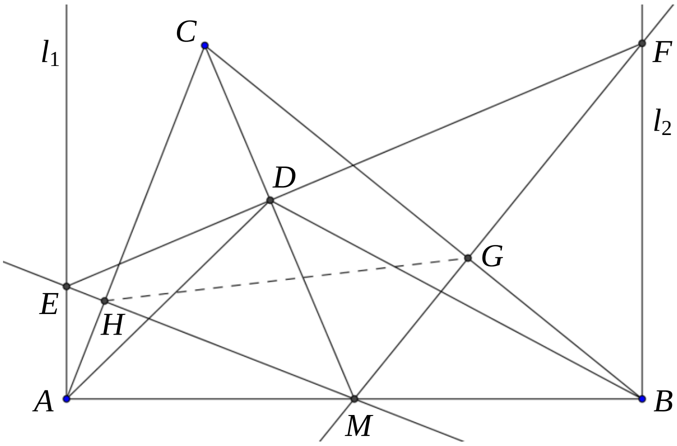
Solution:
Let H, G be the points of intersection of ME, MF with AC, BC respectively. From the similarity of triangles ΔMHA and ΔMAE we get $\frac{MH}{MA}=\frac{MA}{ME}$, thus

$$MA^2=MH\cdot ME. \qquad (1)$$

Similarly, from the similarity of triangles ΔMBG and ΔMFB we get $\frac{MB}{MF}=\frac{MG}{MB}$, thus

$$MB^2=MF\cdot MG. \qquad (2)$$

Since $MA=MB$, from (1), (2), we conclude that the points E, H, G, F are concyclic.



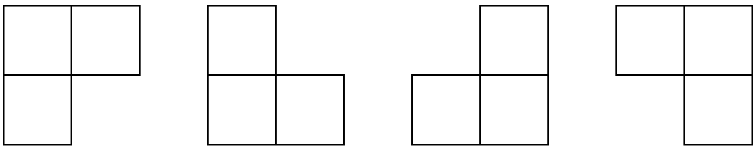
Therefore, we get that $\angle FEH = \angle FEM = \angle HGM$. Also, the quadrilateral $CHMG$ is cyclic, so $\angle CMH = \angle HGC$. We have

$$\angle FEH + \angle CMH = \angle HGM + \angle HGC = 90^\circ,$$

thus $CM \perp EF$. Now, from the cyclic quadrilaterals $FDMB$ and $EAMD$, we get that $\angle DFM = \angle DBM$ and $\angle DEM = \angle DAM$. Therefore, the triangles ΔEMF and ΔADB are similar, so $\angle ADB = \angle EMF$.

Problem 4.

An L-figure is one of the following four pieces, each consisting of three unit squares:



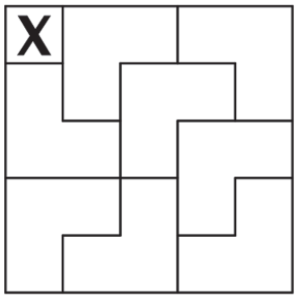
A 5×5 board, consisting of 25 unit squares, a positive integer $k \leq 25$ and an unlimited supply L-figures are given. Two players, ***A*** and ***B***, play the following game: starting with ***A*** they alternatively mark a previously unmarked unit square until they marked a total of k unit squares.

We say that a placement of L-figures on unmarked unit squares is called ***good*** if the L-figure do not overlap and each of them covers exactly three unmarked unit squares of the board. ***B*** wins if every ***good*** placement of L-figures leaves uncovered at least three unmarked unit squares. Determine the minimum value of k for which ***B*** has a winning strategy.

Solution:

We will show that player ***A*** wins if k = 1, 2, 3, but player ***B*** wins if $k = 4$. Thus the smallest k for which ***B*** has a winning strategy exists and is equal to 4.

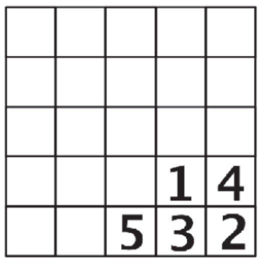
If $k = 1$, player ***A*** marks the upper left corner of the square and then fills it as follows.



If $k=2$, player ***A*** marks the upper left corner of the square. Whatever square player ***B*** marks, then player ***A*** can fill in the square in exactly the same pattern as above except that he doesn't put the L-figure which covers the marked square of ***B***. Player ***A*** wins because he has left only two unmarked squares uncovered.

For $k=3$, player ***A*** wins by following the same strategy. When he has to mark a square for the second time, he marks any yet unmarked square of the L-figure that covers the marked square of ***B***.

Let us now show that for $k=4$ player ***B*** has a winning strategy. Since there will be 21 unmarked squares, player ***A*** will need to cover all of them with seven L-figures. We can assume that in his first move, player ***A*** does not mark any square in the bottom two rows of the chessboard (otherwise just rotate the chessboard). In his first move player ***B*** marks the square labeled 1 in the following figure.



If player ***A*** in his next move does not mark any of the squares labeled 2, 3 and 4 then player ***B*** marks the square labeled 3. Player ***B*** wins as the square labeled 2 is left unmarked but cannot be covered with an L-figure.

If player ***A*** in his next move marks the square labeled 2, then player ***B*** marks the square labeled 5. Player ***B*** wins as the square labeled 3 is left unmarked but cannot be covered with an L-figure.

Finally, if player ***A*** in his next move marks one of the squares labeled 3 or 4, player ***B*** marks the other of these two squares. Player ***B*** wins as the square labeled 2 is left unmarked but cannot be covered with an L-figure.

Since we have covered all possible cases, player ***B*** wins when $k=4$.